A B–Spline Approach to Hermite Subdivision

U. Schwanecke and B. Jüttler

Abstract. We present a new approach to Hermite subdivision schemes. It is based on the observation that a sequence of second order Hermite data define a unique interpolating cubic C^1 spline. The B-Spline form of this interpolating spline leads to a stationary nonuniform subdivision scheme with 4 different subdivision rules for the control points. We construct a generalized 4–point scheme which leads to a new family of C^2 Hermite subdivision schemes.

§1. Introduction

Starting from an initial sequence $\{h_i^{(0)}\}_{i\in\mathbb{Z}}$ of second order Hermite elements (i.e. vectors containing function values and associated first derivatives), a Hermite subdivision scheme (cf. [4,5,6,7]) of order two recursively generates finer sequences $\{h_i^{(k)}\}_{i\in\mathbb{Z}}$ of Hermite elements associated with the dyadic points $\{t_i^{(k)} = i\,2^{-k}\}_{i\in\mathbb{Z}}$. The refinement is based on two rules,

$$h_{2i}^{(k+1)} = \sum_{j=0}^{m} A_j^{(k)} h_{i+j}^{(k)}, \quad h_{2i+1}^{(k+1)} = \sum_{j=0}^{m} B_j^{(k)} h_{i+j}^{(k)}, \quad k = 0, 1, 2, \ldots, \qquad (1)$$

where the matrix masks $\mathbf{A}^{(k)} = \{A_0^{(k)}, \ldots, A_m^{(k)}\}$, $\mathbf{B}^{(k)} = \{B_0^{(k)}, \ldots, B_m^{(k)}\}$ of the scheme consist of real 2×2 matrices $A_j^{(k)}, B_j^{(k)}$ depending on the subdivision level k. Merrien [7] considered Hermite–type 2–point–schemes (i.e. with $m = 1$), generating C^1 functions. By introducing an auxiliary point subdivision scheme, Dyn and Levin [4,5] analyzed stationary Hermite–interpolatory subdivision schemes of arbitrary order. Using this approach, Kuijt [6] constructed several C^2 Hermite interpolatory subdivision schemes of order two. Kuijt derived the refinement rules by considering the polynomials interpolating neighboring Hermite elements, and sampling Hermite data from them.

By considering the interpolating splines associated with the Hermite elements, this paper introduces a new approach to Hermite subdivision. We analyze the smoothness of the limit function, and present a family of C^2 Hermite subdivision schemes generalizing the 4-point scheme [3]. This spline–based approach can be generalized to Hermite elements of arbitrary order.

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§2. Spline Subdivision Schemes

At each subdivision level k, the Hermite data $\{h_i^{(k)}\}_{i\in\mathbb{Z}}$ define a unique interpolating cubic C^1 spline, having the B-spline representation

$$X^{(k)}(t) = \sum_i p_i^{(k)} N_{i,4}(t) \text{ with knots } T^{(k)}=(\ldots,\underbrace{t_i^{(k)},t_i^{(k)}}_{2\times},\underbrace{t_{i+1}^{(k)},t_{i+1}^{(k)}}_{2\times},\ldots). \quad (2)$$

The control points $p_i^{(k)} \in \mathbb{R}$ are associated with the Greville–abscissas (see e.g. [8]) $\xi_{2i}^{(k)} = t_i^{(k)} - \frac{1}{3\,2^k}$ and $\xi_{2i+1}^{(k)} = t_i^{(k)} + \frac{1}{3\,2^k}$, forming a nonuniform sequence. Control points and Hermite elements are related by the transformations $(p_{2i}^{(k)}, p_{2i+1}^{(k)})^\top = H^{(k)} h_i^{(k)}$ and $h_i^{(k)} = (H^{(k)})^{-1}(p_{2i}^{(k)}, p_{2i+1}^{(k)})^\top$, where

$$H^{(k)} = \begin{pmatrix} 1 & -\frac{1}{3\,2^k} \\ 1 & \frac{1}{3\,2^k} \end{pmatrix}, \qquad \left(H^{(k)}\right)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -3\,2^{k-1} & 3\,2^{k-1} \end{pmatrix}. \quad (3)$$

Clearly, the spline function $X^{(k)}$ can be represented with respect to the refined knot vector $T^{(k+1)}$. Knot insertion leads to the following 4 refinement rules for the B-Spline control points:

$$\begin{aligned} \tilde{p}_{4i}^{(k+1)} &= \tfrac{3}{4}p_{2i}^{(k)} + \tfrac{1}{4}p_{2i+1}^{(k)}, & \tilde{p}_{4i+2}^{(k+1)} &= \tfrac{1}{8}p_{2i}^{(k)} + \tfrac{5}{8}p_{2i+1}^{(k)} + \tfrac{2}{8}p_{2i+2}^{(k)}, \\ \tilde{p}_{4i+1}^{(k+1)} &= \tfrac{1}{4}p_{2i}^{(k)} + \tfrac{3}{4}p_{2i+1}^{(k)}, & \tilde{p}_{4i+3}^{(k+1)} &= \tfrac{2}{8}p_{2i+1}^{(k)} + \tfrac{5}{8}p_{2i+2}^{(k)} + \tfrac{1}{8}p_{2i+3}^{(k)}. \end{aligned} \quad (4)$$

The affine combinations (4) describe a stationary nonuniform (due to the nonuniform Greville abscissas) subdivision scheme for the B-Spline control points. This scheme generalizes the splitting step of a binary uniform subdivision scheme. The sequence of control polygons converges to the C^1 limit function $X^{(0)}$. Generalizing (4) leads to the notion of a spline subdivision scheme:

Definition 1. *A* spline subdivision scheme $S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ *with the coefficient masks* $\mathbf{a}^h = (a_0^h, \ldots, a_{2m+1}^h)$, *generating a sequence of cubic* C^1 *spline functions* $X^{(k)}(t)$, *is given by the four subdivision rules*

$$p_{4i+h}^{(k+1)} = \sum_{j=0}^{2m+1} a_j^h p_{2i+j}^{(k)}, \quad h = 0,1,2,3, \qquad k = 0,1,2,\ldots. \quad (5)$$

With the help of the transformations (3), the matrix masks of Hermite subdivision schemes (1) can be transformed into the coefficient masks of spline subdivision scheme (5), thus motivating the following definition.

Definition 2. *A Hermite scheme is said to be* stationary *if the matrices* $A_j := H^{(k+1)} A_j^{(k)} \left(H^{(k)}\right)^{-1}$, $B_j := H^{(k+1)} B_j^{(k)} \left(H^{(k)}\right)^{-1}$, *are constant for all* k ($j = 0, \ldots, m$). *The coefficients* $\mathbf{a}^h$ *of the associated spline subdivision scheme are obtained from*

$$A_j = \begin{pmatrix} a_{2j}^0 & a_{2j+1}^0 \\ a_{2j}^1 & a_{2j+1}^1 \end{pmatrix}, \quad B_j = \begin{pmatrix} a_{2j}^2 & a_{2j+1}^2 \\ a_{2j}^3 & a_{2j+1}^3 \end{pmatrix}.$$

Consequently, every stationary Hermite subdivision scheme $S(\mathbf{A}^{(k)}, \mathbf{B}^{(k)})$ is equivalent to a spline subdivision scheme $S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$.

Note that a spline subdivision scheme can also be seen as a special matrix subdivision scheme (see [1]) acting on vectors of 2 consecutive control points.

§3. Convergence Analysis

In the sequel we generalize the approach introduced by Dyn, Gregory and Levin [2] to the spline subdivision case. Consider a spline subdivision scheme $S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ on the finite domain $[0, n] \in \mathbb{R}$. The scheme is well–defined on this domain, for all $k \geq 0$, if the associated Hermite elements at stage k are defined on the set $\{t_i^{(k)} | i \in Z_k\}$, where

$$Z_k = \{0, 1, \ldots, 2^k n + n_1\}, \quad n_1 = \begin{cases} 2m-1 & \text{if} \quad A_m \neq 0 \\ 2m-2 & \text{if} \quad A_m = 0 \end{cases}.$$

The spline function (2) has the knots

$$T^{(k)} = (t_{-1}^{(k)}, t_0^{(k)}, t_0^{(k)}, t_1^{(k)}, t_1^{(k)}, \ldots, t_{2^k n + n_1}^{(k)}, t_{2^k n + n_1}^{(k)}, t_{2^k n + n_1 + 1}^{(k)})$$

and the control points $(p_i^{(k)})_{i=0,\ldots,2(2^k n + n_1)+1}$.

Consider an interval $I_i^{(k)} = [t_i^{(k)}, t_{i+1}^{(k)}]$ at the k-th subdivision step. The control points which govern the future behavior of the process in this interval are gathered in the vector $\mathbf{p}_{i,k} = (p_{2i}^{(k)}, \ldots, p_{2(i+n_1+1)+1}^{(k)})^\top$.

The control point vectors $\mathbf{p}_{2i,k+1}, \mathbf{p}_{2i+1,k+1}$ at the subdivision level $k+1$, associated with the subintervals $I_{2i}^{(k+1)}$ and $I_{2i+1}^{(k+1)}$, are obtained from $\mathbf{p}_{i,k}$ by two linear transformations,

$$\mathbf{p}_{2i,k+1} = G_0 \, \mathbf{p}_{i,k} \quad \text{and} \quad \mathbf{p}_{2i+1,k+1} = G_1 \, \mathbf{p}_{i,k} \tag{6}$$

with $G_0 = G\binom{1 \ldots M-2}{1 \cdots M-2}$ and $G_1 = G\binom{3 \cdots M}{1 \ldots M-2}$, where $G\binom{i_1 \cdots i_p}{j_1 \cdots j_p}$ is the matrix comprised of the elements of G, at rows $i_1 < \cdots < i_p$ and columns $j_1 < \cdots < j_p$. These linear transformations are expressed as submatrices of the $M \times M$ *generator matrix* G, where $M = 2(n_1 + 3)$. If $A_m \neq 0$, then $M = 4(m+1)$ and we get the generator matrix

$$G = \begin{pmatrix} A_0 & \cdots & \cdots & A_m & 0 & \cdots & \cdots & 0 \\ B_0 & \cdots & \cdots & B_m & 0 & \cdots & \cdots & 0 \\ 0 & A_0 & \cdots & \cdots & A_m & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & \cdots & B_m & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & A_0 & \cdots & \cdots & A_m & 0 \\ 0 & \cdots & 0 & B_0 & \cdots & \cdots & B_m & 0 \end{pmatrix}. \tag{7}$$

Otherwise, if $A_m = 0$, $M = 4m + 2$ and the generator matrix is as above but with the last two rows and columns deleted.

3.1. Continuity

The following necessary condition is analogous to [2, Prop. 2.3]. Alternatively it can be formulated using the eigenstructure of the masks of the associated matrix subdivision scheme, cf. [1].

Proposition 3 (Affine invariance). *A necessary condition for the uniform convergence of the spline subdivision process to a continuous nonzero limit function on* $[0,n]$*, for arbitrary initial data, is that* $\sum_{j=0}^{2m+1} a_j^h = 1$ *for all* $h = 0,1,2,3$.

In order to analyze the convergence to a continuous limit function, we examine the *difference scheme* $\Delta S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ generating the differences $\Delta_i^{(k)} = p_{i+1}^{(k)} - p_i^{(k)}$. If the necessary condition of Proposition 3 is satisfied, then this process can again be described with the help of another generator matrix

$$C = E_M\, G\, (E_M)^{-1} \begin{pmatrix} 1 \ldots M-1 \\ 1 \ldots M-1 \end{pmatrix} \tag{8}$$

which is obtained using the upper triangular matrices $E_M = (-\delta_{i,j} + \delta_{i+1,j})$ and $(E_M)^{-1} = (-\sum_{h=0}^{M-1} \delta_{i+h,j})$, cf. [2, Prop. 3.2]. The $M-3$ differences governing the future behavior of the process in the interval $I_i^{(k)}$ are again collected in a vector $\mathbf{\Delta}_{i,k} = [\Delta_{2i}^{(k)}, \ldots, \Delta_{2(i+n_1+1)}^{(k)}]^\top$. The analogues of the transformation (6) are

$$\mathbf{\Delta}_{2i,k+1} = C_0 \mathbf{\Delta}_{i,k} \quad \text{and} \quad \mathbf{\Delta}_{2i+1,k+1} = C_1 \mathbf{\Delta}_{i,k} \tag{9}$$

where $C_0 = C\binom{1\cdots M-3}{1\cdots M-3}$ and $C_1 = C\binom{3\cdots M-1}{1\cdots M-3}$. Note that the row and column ranges of the sub–matrices C_0, C_1 are different from those in [2], as we analyze a difference process with 4 (rather than 2) rules. We get (cf. [2, Theorem 3.1])

Theorem 4. *Let the spline subdivision process satisfy the necessary convergence condition of Proposition 3. Then the following are equivalent:*

(*i*) *The spline subdivision process* $S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *converges uniformly to a continuous limit function on* $[0,n]$ *for arbitrary initial data.*

(*ii*) *The difference process* $\Delta S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *is contracting, i.e. it converges uniformly to zero on* $[0,n]$ *for arbitrary initial data.*

(*iii*) *There exists an integer* $L > 0$ *and a real number* $0 \le \alpha < 1$ *such that* $\|C_{i_1} \cdots C_{i_L}\|_\infty \le \alpha$, $\forall\, i_j \in \{0,1\}$ *and* $j = 1,\ldots,L$.

In the sequel we have to analyze other point processes with four different refinement rules. The continuity of the limit function can then be analyzed in an analogous way, where the generator matrix is obtained as in (8).

3.2. Derivative process

In order to investigate the differentiability of the limit function f, we analyze the first derivative of the cubic C^1 splines (2). Clearly, we obtain a sequence of quadratic C^0 splines with knots $T^{(k)}$, see Figure 1. If the necessary condition of Prop. 3 is satisfied, then the quadratic splines are generated by another spline subdivision scheme, again with four different rules for the control points. This scheme will be called the derivative scheme $\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$.

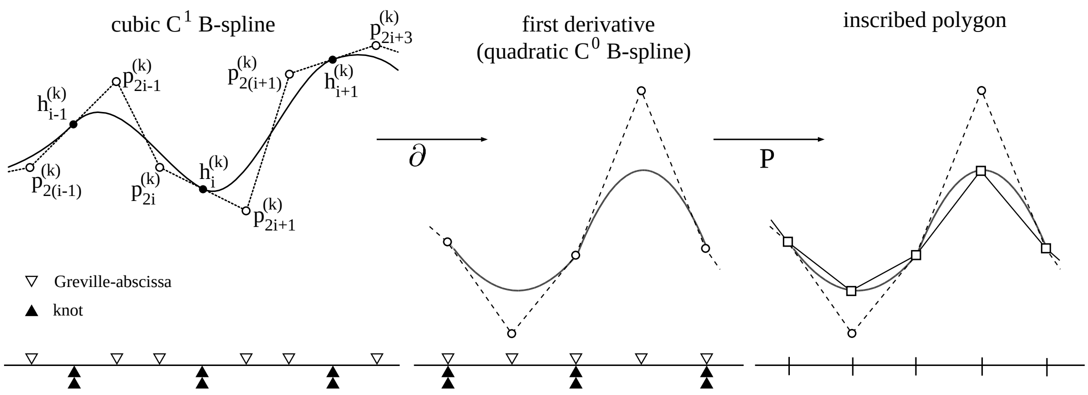

Fig. 1. Derivative scheme and inscribed polygon process.

Proposition 5. *If the derivative process* $\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ *converges uniformly to* $d \in C[0, n]$*, then the spline subdivision scheme* $S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ *converges uniformly to* $f \in C^1[0, n]$*, and* $f' = d$*.*

Using similar techniques as in Section 3.1, we define control point vectors and a generator matrix $\mathcal{D}$. We omit the details, giving only the main result:

Proposition 6. *The derivative scheme* $\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ *has the* $(M-1) \times (M-1)$ *generator matrix*

$$\mathcal{D} = 2E_M^1 \; G \; (E_M^1)^{-1} \binom{1 \ldots M-1}{1 \ldots M-1}$$

with $E_M^1 = \operatorname{diag}(1, 2, 1, \ldots, 2)\, E_M$ *and* $(E_M^1)^{-1} = (E_M)^{-1} \operatorname{diag}(1, \frac{1}{2}, 1, \ldots, \frac{1}{2})$.

The continuity of the limit function generated by the derivative scheme can now easily be analyzed as in Section 3.1 by discussing the associated difference scheme $\Delta\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$. This leads to criteria for C^1 continuity of Hermite subdivision schemes.

3.3. C^k convergence analysis via inscribed polygons

In order to examine higher order continuity, we inscribe a polygon into the quadratic C^0 spline and analyze the resulting subdivision scheme, called the inscribed polygon process $P\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$. More precisely, at the subdivision level k, we consider the piecewise linear function with the vertices $(\,t_i^{(k+1)}, \dot{X}^{(k)}(t_i^{(k+1)})\,)$, see Figure 1.

Proposition 7. *The inscribed polygon process* $P\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$ *of the derivative scheme has the generator matrix* $P = L_{M-1}\; \mathcal{D}\; (L_{M-1})^{-1}$ *which is obtained using auxiliary the* $(M-1) \times (M-1)$ *matrices*

$$L_{M-1} = \begin{pmatrix} 1 & & & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & & & & 1 \end{pmatrix}, \quad (L_{M-1})^{-1} = \begin{pmatrix} 1 & & & & & \\ -\frac{1}{2} & 2 & -\frac{1}{2} & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & -\frac{1}{2} & 2 & -\frac{1}{2} \\ & & & & & 1 \end{pmatrix}$$

The derivative and the inscribed polygon processes are equivalent:

Lemma 8. *The derivative process* $\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *converges uniformly to a continuous limit function* f *on* $[0,n]$ *if and only if the inscribed polygon process* $P\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *converges uniformly to a continuous function* $g \in C[0,n]$. *Moreover,* $f = g$.

Proof: This equivalence is due to the convex hull property of B-splines, and to approximation properties of interpolating quadratic C^0 splines. □

Using the inscribed polygon process, we are now able to discuss the convergence of the spline subdivision scheme to limit functions with higher order differentiability. We simply have to analyze the divided difference processes $D^\nu P\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ (see [2, Theorem 4.2]) of the inscribed polygons, as follows.

Theorem 9. *If the* l*-th order divided difference scheme of the inscribed polygon process* $P\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *exists and converges uniformly to* $f_l \in C[0,n]$, *then also the divided difference processes* $D^\nu P\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *exist and converge uniformly to* $f_\nu \in C^{l-\nu}[0,n]$ *for* $\nu = 0,1,\ldots,l$, *and* $f_0^{(\nu)} = f_\nu$. *Hence, the spline subdivision scheme* $S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ *converges uniformly to* $g \in C^{l+1}[0,n]$ *with* $g^{(\nu+1)} = f_\nu$.

For instance, in order to prove that the limit function generated by the spline subdivision scheme is C^2, the difference process $\Delta DP\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ of the divided difference scheme has to be shown to be contractive, analogously to Section 3.1. The divided difference scheme $DP\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$ has the generator matrix $D = 2E_{M-1}\,P\,(E_{M-1})^{-1}\binom{1\ldots M-2}{1\ldots M-2}$. From this matrix we get the generator matrix $C^* = E_{M-2}\,D\,(E_{M-2})^{-1}\binom{1\ldots M-3}{1\ldots M-3}$ of the associated difference scheme $\Delta DP\partial S(\mathbf{a}^0,\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3)$. In order to guarantee a C^2 limit function, the matrix norms $\|C^*_{i_1}\cdots C^*_{i_L}\|_\infty$, $\forall\ i_j \in \{0,1\}$ and $j = 1,\ldots,L$ have to be less than 1 for some L, where $C^*_0 = C^*\binom{1\ldots M-5}{1\ldots M-5}$, and $C^*_1 = C^*\binom{3\ldots M-3}{1\ldots M-5}$.

§4. A Generalized 4–Point Scheme

Based on a geometric construction, Dyn and Levin [3] derived a family of interpolating 4-point schemes. This family can also be obtained from an optimization–based approach, as follows. Let the subdivision scheme generate a sequence of piecewise linear functions $Y^{(k)}$ with knots $t_i^{(k)}$ and control points $q_i^{(k)}$. In order to derive the refinement rules, we replace one segment of $Y^{(k)}$ with two new ones (shown as dashed lines in Figure 2, left), subject to C^0 boundary conditions. The new vertex $q_{2i+1}^{(k+1)}$ is placed by minimizing the jumps of the first derivatives between new and old polygons. In fact, minimizing the weighted linear combination

$$F(q_{2i+1}^{(k+1)}) = \tfrac{1-4\omega}{4}\,[\,\dot Y_-^{(k+1)}(t_{2i+1}^{(k+1)}) - \dot Y_+^{(k+1)}(t_{2i+1}^{(k+1)})\,]^2 \\ + 2\omega\,[\,\dot Y_-^{(k)}(t_i^{(k)}) - \dot Y_+^{(k+1)}(t_i^{(k)})\,]^2 + 2\omega\,[\,\dot Y_+^{(k+1)}(t_{i+1}^{(k)}) - \dot Y_-^{(k)}(t_{i+1}^{(k)})\,]^2$$

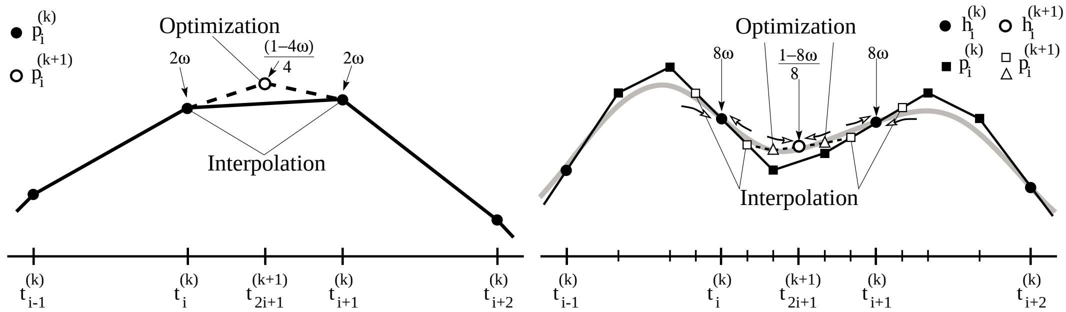

Fig. 2. Weights of objective functions in the point (left) and spline (right) case.

of squared differences of derivatives produces exactly the refinement rules of the interpolating 4-point scheme.

This approach can be generalized to the spline case. In order to derive the refinement rules we replace one segment of $X^{(k)}$ with two new ones (shown with dashed control polygons in Figure 2, right), subject to C^1 boundary conditions. The inner new control points $p_{4i+2}^{(k+1)}$, $p_{4i+3}^{(k+1)}$ are placed by minimizing the jumps of the second derivatives between new and old splines. Minimizing the weighted linear combination

$$F(p_{4i+2}^{(k+1)}, p_{4i+3}^{(k+1)}) = \tfrac{1-8\omega}{8}\,[\,\ddot{X}_-^{(k+1)}(t_{2i+1}^{(k+1)}) - \ddot{X}_+^{(k+1)}(t_{2i+1}^{(k+1)})\,]^2$$
$$+\,8\omega\,[\,\ddot{X}_-^{(k)}(t_i^{(k)}) - \ddot{X}_+^{(k+1)}(t_i^{(k)})\,]^2 + 8\omega\,[\,\ddot{X}_i^{(k+1)}(t_{i+1}^{(k)}) - \ddot{X}_+^{(k)}(t_{i+1}^{(k)})\,]^2$$

of the squared differences of the second derivatives gives the refinement masks

$$\mathbf{a}^0 = (0, 0, \tfrac{3}{4}, \tfrac{1}{4}, 0, 0, 0, 0), \qquad \mathbf{a}^1 = (0, 0, \tfrac{1}{4}, \tfrac{3}{4}, 0, 0, 0, 0),$$
$$\mathbf{a}^2 = (0, \tfrac{1}{8} + \omega, -\tfrac{1}{8} - 2\omega, \tfrac{3}{4} + 3\omega, \tfrac{3}{8} - 3\omega, -\tfrac{1}{4} + 2\omega, \tfrac{1}{8} - \omega, 0),$$
$$\mathbf{a}^3 = (0, \tfrac{1}{8} - \omega, -\tfrac{1}{4} + 2\omega, \tfrac{3}{8} - 3\omega, \tfrac{3}{4} + 3\omega, -\tfrac{1}{8} - 2\omega, \tfrac{1}{8} + \omega, 0),$$

see (5). In order to analyze C^2 continuity of the limit function, we compute the generator matrix of the difference process $\Delta DP\partial S(\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3)$, which has to be contractive. Using the techniques of Section 3.1, we estimate the C^2 convergence range of the parameter ω, see Figure 3.

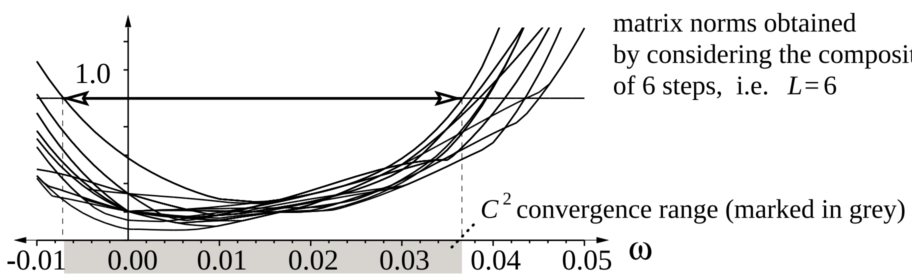

Fig. 3. Estimating the C^2 convergence range of the generalized 4 point scheme.

Two limit functions interpolating three given Hermite elements have been drawn in Figure 4 (left). The functions have been generated with the help of Merrien's C^1 scheme ($\alpha = 0.2$, dashed curve, cf. [4,7]), and using the generalized 4-point scheme ($\omega = 0.015$, solid curve). As can clearly be seen from the associated first derivatives (right), the generalized 4-point scheme produces a C^2 limit function.

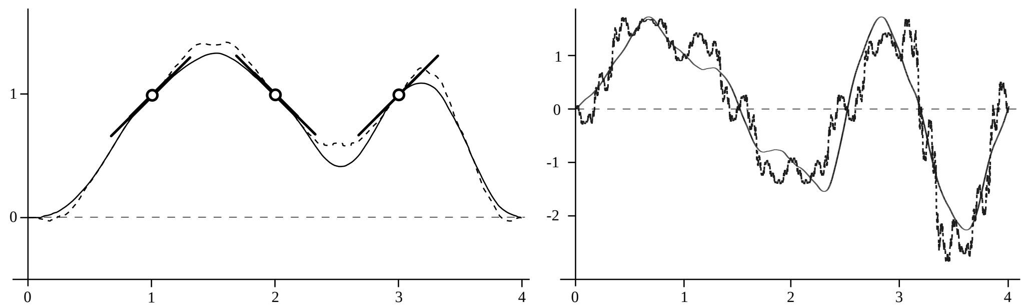

Fig. 4. Interpolatory limit functions (left) and derivatives (right).

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